TRIPLE PRODUCTS AND YANG–BAXTER EQUATION (II): ORTHOGONAL AND SYMPLECTIC TERNARY SYSTEMS

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Abstract

We generalize the result of the preceeding paper and solve the Yang–Baxter equation in terms of triple systems called orthogonal and symplectic ternary systems. In this way, we found several other new solutions.

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1. Introduction and Summary of Results

Let V be a N-dimensional vector space with a bi-linear non-degenerate form (or inner product) $< x|y >$ for $x, y \epsilon V$. Let $e_1, e_2, \ldots, e_N$ be a basis of V and set

$$< e_j|e_k > = g_{jk} \tag{1.1}$$

with its inverse g^{jk} satisfying

$$g^{jk} g_{k\ell} = \delta^j_\ell \quad . \tag{1.2}$$

We raise or lower indices, as usual, by g^{jk} or g_{jk}. For example, we set

$$e^j = g^{jk} e_k \tag{1.3}$$

so that we have

$$< e^j|e_k > = \delta^j_k \tag{1.4}$$

as well as

$$e_j < e^j|x > = < x|e_j > e^j = x \quad . \tag{1.5}$$

In the preceeding paper[1)] which we refer to hereafter as I, we have rewritten the Yang–Baxter (Y–B) equation

$$\begin{aligned} &R^{b'a'}_{a_1b_1}(\theta) R^{c'a_2}_{a'c_1}(\theta') R^{c_2b_2}_{b'c'}(\theta'') \\ &\qquad = R^{c'b'}_{b_1c_1}(\theta'') R^{c_2a'}_{a_1c'}(\theta') R^{b_2a_2}_{a'b'}(\theta) \end{aligned} \tag{1.6}$$

with

$$\theta' = \theta + \theta'' \tag{1.7}$$

as a triple product equation

$$\begin{aligned} &[v, [u, e_j, z]_{\theta'}, [e^j, x, y]_\theta]_{\theta''} \\ &\qquad = [u, [v, e_j, x]_{\theta'}, [e^j, z, y]_{\theta''}]_\theta \quad , \end{aligned} \tag{1.8}$$

provided that the scattering matrix element $R^{ab}_{cd}(\theta)$ satisfies the symmetry condition

$$R^{ab}_{cd}(\theta) = R^{ba}_{dc}(\theta) \tag{1.9}$$

or equivalently

$$< u|[z,x,y]_\theta > = < z|[u,y,x]_\theta > \quad . \tag{1.10}$$

Here, the θ-dependent triple product $[x,y,z]_\theta$ has been defined by

$$R^{ab}_{cd}(\theta) = < e^a|[e^b, e_c, e_d]_\theta > \quad . \tag{1.11}$$

The more general case without assuming Eq. (1.9) will be discussed in section 6. In I, we have solved the Y–B equation for two cases of $N = 4$ and 8, corresponding to quaternionic and octonionic triple products. The purpose of this note is to generalize the method for more general cases of any orthogonal and symplectic ternary systems satisfying a condition to be specified shortly. To be definite, we shall first give axioms for these systems below. Suppose that the vector space V possesses a $\underline{\theta\text{-independent}}$ triple product

$$xyz \ : \ V \otimes V \otimes V \rightarrow V \tag{1.12}$$

as well as the non-degenerate bi-linear form $< x|y >$. Let ε be a constant assuming value of either $\varepsilon = +1$ or $\varepsilon = -1$. Our fundamental ansatz is then that they satisfy axioms:

(i) $< y|x > = \varepsilon < x|y >$ (1.13a)

(ii) $xyz + \varepsilon \ yxz = 0$ (1.13b)

(iii) $xyz + \varepsilon \ xzy = 2\lambda < y|z > x - \lambda < x|y > z - \lambda < z|x > y$ (1.13c)

(iv) $uv(xyz) = (uvx)yz + x(uvy)z + xy(uvz)$ (1.13d)

(v) $< uvx|y > = - < x|uvy >$ (1.13e)

for $u, v, x, y, z \, \epsilon \, V$, where λ in Eq. (1.13c) is a constant. Then, the case of $\epsilon = +1$ defines the orthogonal ternary system (OTS) as in I, while the other case of $\epsilon = -1$ is called by Yamaguchi and Asano[2)] to be the symplectic ternary system (STS). Both OTS and STS may be regarded as special cases of more general triple systems discussed by many authors[3)–9)], whose studies will be left, however, in the future.

Before going into further details, we note that the last postulate Eq. (1.13e) is actually a consequence of other postulates Eqs. (1.13a)–(1.13d), provided that we have $\lambda \neq 0$ and

Dim $V \geq 2$ for $\varepsilon = 1$. However, since we consider sometime the special case of $\lambda = 0$, we added it as an extra postulate here. To show it, we first introduce the notion of a derivation

$$D \; : \; V \to V \tag{1.14}$$

of the triple system to be a linear transformation in V satisfying

$$D(xyz) = (Dx)yz + x(Dy)z + xy(Dz) \quad . \tag{1.15}$$

Applying D to both sides of Eq. (1.13c), we find then an identity

$$\begin{aligned} &2\lambda\{< Dy|z > + < y|Dz >\}x \\ &\qquad - \lambda\{< z|Dx > \; + \; < Dz|x >\}y \; - \lambda\{< Dx|y > \; + \; < x|Dy >\}z = 0 \quad . \end{aligned}$$

Suppose $\lambda \neq 0$, and set $z = x$. For $\varepsilon = -1$, this immediatley gives

$$< x|Dy > = - < Dx|y > \tag{1.16}$$

as has already been observed by Yamaguchi and Asano[2)]. For the other case of $\varepsilon = +1$, Eq. (1.16) will also follow, provided that we have Dim $V \geq 2$ which we will assume hereafter. Next, if we introduce the left multiplication operator $L_{x,y} \; : \; V \to V$ by

$$L_{x,y}z = xyz$$

then Eq. (1.13d) implies that $D = L_{u,v}$ is a derivation of the system so that Eq. (1.13e) will follow readily from Eq. (1.16). In passing, Eq. (1.13d) can be rewritten as a Lie equation

$$[L_{u,v}, L_{x,y}] = L_{uvx,y} + L_{x,uvy} = -L_{xyu,v} - L_{u,xyv} \tag{1.17}$$

which is equivalent to

$$\begin{aligned} uv(xyz) - xy(uvz) &= (uvx)yz + x(uvy)z \\ &= -(xyu)vz - u(xyv)z \quad . \end{aligned} \tag{1.18}$$

For the octonionic triple product corresponding to $\varepsilon = 1$ and $N =$ Dim $V = 8$, Eq. (1.17) defines a so(8) Lie algebra, though V is the 8–dimensional module of the so(7).

Next, we introduce the second triple product

$$[x, y, z] \ : \ V \otimes V \otimes V \to V \tag{1.19}$$

by

$$[x, y, z] = xyz - \lambda < y|z > x + \lambda < z|x > y \quad . \tag{1.20}$$

Then, Eqs. (1.13b), (1.13c) and (1.13e) can be restated as the statement that both $[x, y, z]$ and $< w|[x, y, z] >$ are totally antisymmetric for $\varepsilon = 1$ and totally symmetric for $\varepsilon = -1$, respectively, with respect to 3 variables $x,\ y,$ and z or 4-variables $x, y, z,$ and w. However, the derivation property Eq. (1.13d) becomes rather complicated in terms of $[x, y, z]$. We will profitably utilize, in this note, both notations, alternatively depending upon situations.

Let $e_j (j = 1, 2, \ldots, N)$ be a basis of V. Then, g_{jk} defined by Eq. (1.1) satisfies now

$$g_{kj} = \varepsilon \ g_{jk} \quad , \tag{1.20}$$

so that we have

$$e_j \otimes e^j = \varepsilon \ e^j \otimes e_j \quad , \tag{1.21}$$

$$[x, e_j, e^j] = 0 \quad . \tag{1.22}$$

Moreover, in view of Eqs. (1.5) and (1.13e), we can readily see the validity of

$$xye^j \otimes e_j = -e^j \otimes xye_j = -\varepsilon \ e_j \otimes xye^j \tag{1.23}$$

which will be used often in what follows.

We organize our paper as follows. In section 2, we will study further consequences of both OTS and STS. Especially, we will first show that another triple product $x \cdot y \cdot z$ given by

$$x \cdot y \cdot z = (xye^j)ze_j - \frac{1}{3} \ \lambda(\epsilon N - 7)xyz \tag{1.24}$$

defines Lie[10] and anti-Lie[11] triple products for $\varepsilon = 1$ and $\varepsilon = -1$, respectively. In section 3, we will solve the Y–B equation in a form of

$$\begin{aligned} [z,x,y]_\theta =& P(\theta)xyz + Q(\theta) < x|y > z \\ &+ R(\theta) < z|x > y + S(\theta) < y|z > x \end{aligned} \tag{1.25}$$

for some functions $P(\theta)$, $Q(\theta)$, $R(\theta)$, and $S(\theta)$ to be determined, assuming that OTS or STS satisfies the additional condition of

$$x \cdot y \cdot z = 0 \tag{1.26}$$

identically. Note the change of orders of variables $x, y,$ and z in $[z,x,y]_\theta$ and $P(\theta)xyz$ in Eq. (1.25). This is necessary in order to accomodate the symmetry condition Eq. (1.10) so that we need only solve Eq. (1.8). Rewriting $[z,x,y]_\theta$ as

$$\begin{aligned} [z,x,y]_\theta =& P(\theta)[x,y,z] + A(\theta) < x|y > z \\ &+ B(\theta) < z|x > y + C(\theta) < y|z > x \end{aligned} \tag{1.27}$$

with

$$A(\theta) = Q(\theta) \quad , \tag{1.28a}$$

$$B(\theta) = R(\theta) - \lambda P(\theta) \quad , \tag{1.28b}$$

$$C(\theta) = S(\theta) + \lambda P(\theta) \quad , \tag{1.28c}$$

the solution of the Y–B equation is found in section 3 to be

$$A(\theta) = \{\lambda - \frac{2a\lambda}{2(a-\lambda)+b\theta}\}\ P(\theta) \quad , \tag{1.29a}$$

$$B(\theta) = (a - \lambda + b\theta)P(\theta) \quad , \tag{1.29b}$$

$$C(\theta) = \big(-\lambda - \frac{2a\lambda}{b\theta}\big)\ P(\theta) \quad , \tag{1.29c}$$

where we have set for simplicity

$$a = \frac{1}{6}\ \lambda(4 - \epsilon N) \tag{1.30}$$

while $P(\theta)$ is an undetermined function of θ, and b is an arbitrary constant.

In section 4, we will discuss various OTS and STS satisfying the condition Eq. (1.26), i.e. $x \cdot y \cdot z = 0$. It will be shown there that for $\varepsilon = 1$ (OTS), both octonionic and Malcev triple products with $N = 8$ and $N = 7$, respectively satisfy the condition. Especially, for the former, the solution Eqs. (1.29) will reproduce the result of I with $\lambda = -3\beta = 3$. However, the quaternionic triple product with $\varepsilon N = 4$ whose solution has been given in I does not satisfy $x \cdot y \cdot z = 0$. With respect to the STS case of $\varepsilon = -1$, we have found six solutions with $N = 2,\ 4,\ 14,\ 20,\ 32,$ and 56. They are intimately related to the Lie algebras $A_2,\ G_2,\ F_4,\ E_6,\ E_7,$ and E_8. Especially, the last solution of $N = 56$ corresponds to the celebrated Freudenthal's triple system[12]. Also, for the simplest case of $N = 2$, we can find more general solutions which are either constant or of trigonometric type, as will be studied in section 5. Finally, we will show in section 6 how to rewrite Eq. (1.6) as a triple product equation without assuming Eqs. (1.9) or (1.10).

2. Orthogonal and Symplectic Ternary Systems

In this section, we will study various consequences of OTS and STS, which will be needed for the solution of the Y–B equation to be given in section 3.

First, we note that xyz given by

$$xyz = \lambda < y|z > x - \lambda < z|x > y$$

or equivalently

$$[x, y, z] = 0$$

satisfies all axioms Eqs. (1.13b)–(1.13e) of OTS and STS when we note Eq. (1.13a). We call such a case to be trivial.

We now prove the following lemma.

Lemma 1

Let V be either OTS or STS. Then, we have

(i) $e_j e^j x = 0 \quad , \quad < e_j | x y e^j > = 0$ (2.1a)

(ii) $x e_j e^j = \lambda(\epsilon N - 1) x$ (2.1b)

(iii) $< u | x v y > = < v | y u x >$ (2.1c)

(iv) $< z | x y e^j > e_j = -\varepsilon \; x y z$ (2.1d)

$< x | e_j y z > e^j = -z x y$ (2.1e)

(v) $< x y e^j | z u e_j > = - < (x y e^j) e_j z | u >$

$= < u | (x y e^j) z e_j > \; -3\lambda < u | x y z >$ (2.1f)

(vi) $(x y e^j) z e_j = z e_j (x y e^j) = (x e^j y) e_j z = -\varepsilon (x y e^j) e_j z + 3\lambda x y z$ (2.1g)

(vi) $< v | (x y e^j) z u > e_j = \epsilon \; x y (u v z) \quad .$ (2.1h)

Proof

Noting Eqs. (1.13b) and (1.21), we calculate

$$e_j e^j x = -\varepsilon \; e^j e_j x = -\varepsilon\varepsilon \; e_j e^j x = -e_j e^j x = 0$$

which proves the 1st relation in Eq. (2.1a). The 2nd relation can be similarly obtained when we use Eqs. (1.13a) and (1.13e). Next, we compute

$$x e_j e^j + \varepsilon \; x e^j e_j = 2\lambda < e_j | e^j > x - \lambda < x | e_j > e^j - \lambda < e^j | x > e_j$$

$$= 2\lambda \; \varepsilon \; N \; x - \lambda x - \lambda x = 2\lambda(\epsilon N - 1) x$$

from Eqs. (1.13a), (1.13c), (1.4) and (1.5). On the other side, we have

$$\varepsilon \; x \; e^j e_j = \varepsilon\varepsilon \; x \; e_j e^j = x e_j e^j$$

in view of Eq. (1.21). This proves Eq. (2.1b). As for (iii), we rewrite

$$< u | x v y > \; = < u | [x, v, y] > \; + \lambda < v | y >< u | x > \; - \lambda < y | x >< u | v > \quad ,$$

$$< v | y u x > \; = < v | [y, u, x] > \; + \lambda < u | x >< v | y > \; - \lambda < x | y >< v | u >$$

and note that $< u | [x, v, y] >$ is totally symmetric for $\varepsilon = 1$ and totally antisymmetric for $\varepsilon = -1$, respectively. Comparing both, this gives Eq. (2.1c).

Next, Eq. (2.1d) is a immediate consequence of Eqs. (1.13e), (1.13a) and (1.5), while Eq. (2.1e) follows then from Eqs. (2.1d) and (2.1c) when we rewrite it

$$< x|e_j yz > e^j = < y|zxe_j > e^j = \varepsilon < y|zxe^j > e_j = -zxy \quad .$$

Replacing $v \to u \to xye^j,\ x \to z$ and $y \to e_j$ in Eq. (2.1c), we find

$$< xye^j|zue_j > = < u|e_j(xye^j)z > = -\varepsilon < u|(xye^j)e_j z > = - < (xye^j)e_j z|u >$$

which gives the 1st relation in Eq. (2.1d). Then, the last relation in Eq. (2.1d) follows from Eq. (2.1g) to be proved below. Now Eq. (1.23) readily gives

$$(xye^j)ze_j = -\varepsilon\ e_j z(xye^j) = ze_j(xye^j)$$

which proves the 1st relation in Eq. (2.1g). In order to show the rest of equations, we calculate

$$\begin{aligned}
(xe^j y)e_j z &= \{-\varepsilon\ xye^j + 2\lambda < e^j|y > x - \lambda < x|e^j > y - \lambda < y|x > e^j\}e_j z \\
&= -\varepsilon(xye^j)e_j z + 2\lambda < e^j|y > xe_j z - \lambda < x|e^j > ye_j z - \lambda < y|x > e^j e_j z \\
&= -\varepsilon(xye^j)e_j z + 2\lambda\ xyz - \lambda\epsilon\ yxz \\
&= -\varepsilon(xye^j)e_j z + 3\lambda\ xyz \\
&= -\varepsilon\{-\varepsilon(xye^j)ze_j + 2\lambda < e_j|z > xye^j \\
&\quad - \lambda < xye^j|e_j > z - \lambda < z|xye^j > e_j\} + 3\lambda\ xyz \\
&= (xye^j)ze_j - 2\lambda\ xyz - \lambda\ xyz + 3\lambda\ xyz \\
&= (xye^j)ze_j
\end{aligned}$$

when we note Eqs. (1.13b), (1.13c), (1.5), (2.1a) and (2.1d). Finally, from Eqs. (2.1c) and (1.13e), we find

$$\begin{aligned}
< v|(xye^j)zu > e_j &= < z|uv(xye^j) > e_j = - < uvz|xye^j > e_j \\
&= < xy(uvz)|e^j > e_j = \varepsilon\ xy(uvz)
\end{aligned}$$

which is Eq. (2.1h). This completes the proof of the Lemma 1. ■

Proposition 1

The new triple product defined by

$$x \cdot y \cdot z = (xye^j)ze_j - \frac{1}{3}\ \lambda(\epsilon N - 7)xyz \quad , \tag{2.2}$$
$$= -(xye_j)e^j z - \frac{1}{3}\ \lambda(\epsilon N - 16)xyz$$

satisfies the following relations:

$$\text{(i)}\ \ x \cdot y \cdot z = -\varepsilon\ y \cdot x \cdot z \quad , \tag{2.3a}$$

$$\text{(ii)}\ \ x \cdot y \cdot z + y \cdot z \cdot x + z \cdot x \cdot y = 0 \quad , \tag{2.3b}$$

$$\text{(iii)}\ \ u \cdot v \cdot (x \cdot y \cdot z) = (u \cdot v \cdot x) \cdot y \cdot z + x \cdot (u \cdot v \cdot y) \cdot z + x \cdot y \cdot (u \cdot v \cdot z) \quad , \tag{2.3c}$$

$$\text{(iv)}\ \ uv(x \cdot y \cdot z) = (uvx) \cdot y \cdot z + x \cdot (uvy) \cdot z + x \cdot y \cdot (uvz) \quad , \tag{2.3d}$$

$$\text{(v)}\ \ u \cdot v \cdot (xyz) = (u \cdot v \cdot x)yz + x(u \cdot v \cdot y)z + xy(u \cdot v \cdot z) \quad , \tag{2.3e}$$

$$\text{(vi)}\ \ < u \cdot v \cdot x | y > = - < x | u \cdot v \cdot y > = - < v | y \cdot x \cdot u > \quad . \tag{2.3f}$$

Especially Eqs. (2.3a)–(2.3c) imply that it defines a Lie[10] or anti-Lie[11] triple system, respectively, for $\varepsilon = 1$ or $\varepsilon = -1$.

Proof

The first relation Eq. (2.3a) is a immediate consequence of Eq. (1.13b).

Next by the derivation relation Eq. (1.13d), we calculate

$$\begin{aligned} ze_j(xye^j) &= (ze_jx)ye^j + x(ze_jy)e^j + xy(ze_je^j) \\ &= (ze_jx)ye^j - \varepsilon(ze_jy)xe^j + \lambda(\epsilon N - 1)xyz \quad . \end{aligned} \tag{2.4}$$

Moreover, we continue

$$\begin{aligned} (ze_jx)ye^j &= \{-\epsilon\ zxe_j + 2\lambda < e_j|x > z - \lambda < z|e_j > x - \lambda < x|z > e_j\}ye^j \\ &= -\varepsilon(zxe_j)ye^j + 2\lambda\ \varepsilon\ zyx - \lambda xyz + \lambda\varepsilon < x|z > ye_je^j \\ &= -(zxe^j)ye_j - 2\lambda\ yzx - \lambda xyz + \lambda^2 < z|x > (\epsilon N - 1)y \\ &= -(zxe^j)ye_j - 3\lambda[x, y, z] - \lambda^2 < y|z > x \\ &\quad + 2\lambda^2 < x|y > z + \lambda^2(\epsilon N - 2) < z|x > y \end{aligned}$$

after some calculation. Then, Eq. (2.4) together with Eq. (2.1g) leads to

$$
\begin{aligned}
&(xye^j)ze_j + (zxe^j)ye_j + (yze^j)xe_j \\
&\qquad = \lambda(\epsilon N - 7)[x, y, z] = \frac{1}{3}\ \lambda(\epsilon N - 7)(xyz + yzx + zxy)
\end{aligned}
$$

which gives Eq. (2.3b).

In order to prove Eqs. (2.3d)–(2.3f), we first set

$$
x * y * z = (xye^j)ze_j = -(xye_j)e^j z + 3\lambda xyz \quad , \tag{2.5}
$$

and calculate

$$
\begin{aligned}
&uv(x * y * z) - (uvx) * y * z - x * (uvy) * z - x * y * (uvz) \\
&\qquad = \{xy(uve^j)\}ze_j + (xye^j)z(uve_j)
\end{aligned}
$$

from Eq. (1.13d). However, the right side of this relation is identically zero because of Eq. (1.23) i.e. $(uve^j) \otimes e_j + e^j \otimes (uve_j) = 0$, so that we find

$$
uv(x * y * z) = (uvx) * y * z + x * (uvy) * z + x * y * (uvz) \quad . \tag{2.6}
$$

Then, Eq. (2.3d) follows readily from Eq. (2.6) and

$$
x \cdot y \cdot z = x * y * z - \frac{1}{3}\ \lambda(\epsilon N - 7)xyz \quad . \tag{2.7}
$$

Next, we note

$$
(uve^j)e_j(xyz) = \{(uve^j)e_jx\}yz + x\{(uve^j)e_jy\}z + xy\{(uve^j)e_jz\} \quad .
$$

Together with Eq. (2.5), this gives

$$
u * v * (xyz) = (u * v * x)yz + x(u * v * y)z + xy(u * v * z) \tag{2.8}
$$

and hence Eq. (2.3e). Similarly, Eq. (2.3c) is a consequence of

$$
u * v * (x * y * z) = (u * v * x) * y * z + x * (u * v * y) * z + x * y * (u * v * z) \tag{2.9}
$$

which results from Eq. (1.13d) as well as

$$(x * y * e^j) \otimes e_j = -e^j \otimes (x * y * e_j) \quad , \tag{2.10a}$$

or equivalently

$$< x * y * u|v > = - < u|x * y * v > \tag{2.10b}$$

which is the analogue of Eq. (1.23). Finally, we can verify Eq. (2.3f) directly. We may note that if $\lambda \neq 0$, then $< u \cdot v \cdot x|y > = - < x|u \cdot v \cdot y >$ is a simple consequence of Eq. (1.16), since $D = L^*_{x,y}$ defined by

$$L^*_{x,y} z = x \cdot y \cdot z$$

is a derivation of the original OTS or STS because of Eq. (2.3e). At any rate, these complete the proof of the Proposition 1.

Proposition 2

We have

$$\begin{aligned} 2(xye^j)(zue_j)v = \varepsilon(x \cdot y \cdot z)uv - (x \cdot y \cdot u)zv \\ + \frac{1}{3}\, \varepsilon\lambda(\epsilon N - 16)(xyz)uv - \frac{1}{3}\, \lambda(\epsilon N - 16)(xyu)zv \quad . \end{aligned} \tag{2.11}$$

Corollary 1

We have

(i) $\varepsilon(xyz)uv - (xyu)zv + \varepsilon(zux)yv - (zuy)xv = 0$

(ii) $\varepsilon(x \cdot y \cdot z)uv - (x \cdot y \cdot u)zv + \varepsilon(z \cdot u \cdot x)yv - (z \cdot u \cdot y)xv = 0$

Proof

We first rewrite Eq. (1.13d) as

$$\begin{aligned} &(uvx)yz + (uvy)zx + (uvz)xy - uv[x, y, z] \\ &= \lambda < y|z > uvx + \lambda < z|x > uvy + \lambda < x|y > uvz \\ &\quad - \lambda < y|uvz > x - \lambda < z|uvx > y - \lambda < x|uvy > z \quad , \end{aligned} \tag{2.12}$$

and let $x \to u \to xye^j,\ y \to e_j,$ and $v \leftrightarrow z$ in Eq. (2.12) to find

$$\begin{aligned}&[(xye^j)zu]e_jv + [(xye^j)ze_j]vu + [(xye^j)zv]ue_j - (xye^j)z[u,e_j,v] \\ &\quad = \lambda < e_j|v > (xye^j)zu + \lambda < v|u > (xye^j)ze_j + \lambda < u|e_j > (xye^j)zv \\ &\quad - \lambda < e_j|(xye^j)zv > u \ - \ \lambda < v|(xye^j)zu > e_j - \lambda < u|(xye^j)ze_j > v \quad .\end{aligned} \tag{2.13}$$

We now rewrite 4-terms in the left side of Eq. (2.13) respectively as

$$\begin{aligned}&[(xye^j)zu]e_jv \\ &\quad = (xye^j)(zue_j)v - 2\lambda(xyu)zv \\ &\quad + \lambda\varepsilon(xyz)uv + \lambda < z|u > (xye^j)e_jv \quad ,\end{aligned} \tag{2.14a}$$

$$\begin{aligned}&[(xye^j)ze_j]vu \\ &\quad = vu[(xye^j)ze_j] - 2\lambda < u|(xye^j)ze_j > v \\ &\quad + \lambda\varepsilon < v|(xye^j)ze_j > u + \lambda\varepsilon < u|v > (xye^j)ze_j \quad ,\end{aligned} \tag{2.14b}$$

$$\begin{aligned}&[(xye^j)zv]ue_j \\ &\quad = -\varepsilon(xye^j)(zve_j)u + 2\lambda\varepsilon(xyv)zu - \lambda(xyz)vu \\ &\quad + 2\lambda(xyu)zv - \lambda xy(vuz) \\ &\quad - \lambda < v|z > (xye^j)e_ju + \lambda\varepsilon < v|(xye^j)ze_j > u \quad ,\end{aligned} \tag{2.14c}$$

$$\begin{aligned}&-(xye^j)z[u,e_j,v] \\ &\quad = -(xye^j)(uve_j)z - 2\lambda\varepsilon\ xy(uvz) + \lambda\varepsilon\ uv(xyz) \\ &\quad - \lambda\varepsilon(xyv)zu + \lambda(xyu)zv - \lambda\varepsilon < uve_j|xye^j > z \quad ,\end{aligned} \tag{2.14d}$$

in the following ways.

First consider Eq. (2.14a). In view of Eq. (1.23), we calculate

$$\begin{aligned}[(xye^j)zu]e_jv &= -(e^jzu)(xye_j)v = \varepsilon(xye_j)(e^jzu)v \\ &= (xye^j)(e_jzu)v = -\varepsilon(xye^j)(ze_ju)v \\ &= -\varepsilon(xye^j)\{-\varepsilon\ zue_j + 2\lambda < e_j|u > z - \lambda < z|e_j > u - \lambda < u|z > e_j\}v \quad ,\end{aligned}$$

which together with Eq. (1.5) leads to Eq. (2.14a). The next relation Eq. (2.14b) is a direct consequence of Eqs. (1.13b) and (1.13c). As for Eq. (2.14c), we first note

$$[(xye^j)zv]ue_j = -(e^j zv)u(xye_j)$$

by Eq. (1.23). Then, Eq. (2.14c) follows after some calculations in a similar way. The same remark also applies for the derivation of Eq. (2.14d).

The right side of Eq. (2.13) can be readily computed from Eqs. (1.5) and (2.1h) to be

$$\begin{aligned} &\lambda\epsilon(xyv)zu + \lambda(xyu)zv - \lambda\varepsilon\ xy(uvz) + \lambda\varepsilon < u|v > (xye^j)ze_j \\ &\quad + \lambda\varepsilon < v|(xye^j)ze_j > u - \lambda < u|(xye^j)ze_j > v \quad . \end{aligned} \tag{2.15}$$

From Eqs. (2.14) and (2.15), we can rewrite Eq. (2.13) in a form of

$$\begin{aligned} &(xye^j)(zue_j)v + (xye^j)(vze_j)u - (xye^j)(uve_j)z \\ &= \varepsilon\ uv[(xye^j)ze_j] - 3\lambda\varepsilon\ uv(xyz) \\ &\quad - \lambda\varepsilon < u|z > (xye^j)e_jv + \lambda < v|z > (xye^j)e_ju \\ &\quad + \lambda < xye^j|zue_j > v - \lambda\varepsilon < xye^j|zve_j > u + \lambda < xye^j|uve_j > z \end{aligned} \tag{2.16}$$

after some calculations. Note that both sides of Eq. (2.16) are manifestly antisymmetric for $\varepsilon = 1$ and symmetric for $\varepsilon = -1$ with respect to the exchange $u \leftrightarrow v$.

Next, we let $u \to v \to z \to u$ in Eq. (2.16) and add it to Eq. (2.16) to obtain

$$\begin{aligned} &2(xye^j)(zue_j)v \\ &\quad = \varepsilon\ uv[(xye^j)ze_j] + \varepsilon\ vz[(xye^j)ue_j] \\ &\quad - 3\lambda\varepsilon\ uv(xyz) - 3\lambda\varepsilon\ vz(xyu) + \lambda\varepsilon < z|v > (xye^j)e_ju \\ &\quad - \lambda < u|v > (xye^j)e_jz + 2\lambda < xye^j|zue_j > v \\ &\quad + 2\lambda < xye^j|uve_j > z - 2\lambda\varepsilon < xye^j|zve_j > u \quad . \end{aligned} \tag{2.17}$$

Using Eqs. (1.13a)–(1.13c), we can still simplify Eq. (2.17) to be

$$\begin{aligned} &2(xye^j)(zue_j)v \\ &\quad = \varepsilon[(xye^j)ze_j]uv - [(xye^j)ue_j]zv - 3\lambda\varepsilon(xyz)uv + 3\lambda(xyu)zv \end{aligned} \tag{2.18}$$

which is equivalent to the desired relation Eq. (2.11).

The relations in Corollary 1 are necessary to satisfy the identity

$$(xye^j)(zue_j)v = -(zue^j)(xye_j)v$$

which is antisymmetric for $x \leftrightarrow z$ and $y \leftrightarrow u$. These can be derived from Eqs. (1.18) and (2.3e) for example, by letting $z \leftrightarrow v$ in Eq. (1.18). We note that the 1st relation in the Corollary can also be rewritten in a more symmetrical form of

$$\varepsilon[x, y, z]uv = [x, y, u]zv + [x, u, z]yv + [u, y, z]xv \quad .$$

This completes the proof of the Proposition 2. ■

Proposition 3

Suppose that $x \cdot y \cdot z$ is trivial in the sense that it is given in a form of

$$x \cdot y \cdot z = \gamma\{< y|z > x - < z|x > y\} \tag{2.19}$$

for a constant γ. Then, we must have $\gamma = 0$ and hence $x \cdot y \cdot z = 0$ identically, unless we have either $\epsilon N = 4$ or $[x, y, z] = 0$ identically. Especially, the last condition implies that the original OTS or STS must be trivial. Conversely, if $[x, y, z] = 0$, then $x \cdot y \cdot z$ satisfies Eq. (2.19) with $\gamma = -\frac{1}{3}\ \lambda^2(\epsilon N - 10)$.

Proof

Since its proof is a bit complicated, we will divide it into the following three 3 steps by assuming the validity of Eq. (2.19).

Step 1

For 4-vectors $w, u, v, z \ \epsilon\ V$, and for a basis vector e_j of V, we have

$$
\begin{aligned}
& w(uve^k)(e_j e_k z) \\
&\quad = \frac{1}{6}\ \lambda\epsilon(\epsilon N - 16)(uvz)e_j w - \frac{1}{6}\ \lambda(\epsilon N - 10) w(uve_j)z \\
&\qquad + 2\lambda\epsilon\ uv(ze_j w) + 2\lambda\epsilon\ w(uvz)e_j + \lambda(wuv)e_j z \\
&\qquad + 2(\lambda)^2 < v|w > ue_j z - (\lambda)^2 < u|v > we_j z - (\lambda)^2\epsilon < u|w > ve_j z \\
&\qquad + \frac{1}{6}\ (\lambda)^2(\epsilon N - 16)\{-2 < z|w > uve_j + \varepsilon < uvz|e_j > w\} \\
&\qquad + \frac{1}{3}\ (\lambda)^2(\epsilon N - 10) < e_j|z > uvw - 2(\lambda)^2\epsilon < w|uvz > e_j \\
&\qquad + \frac{1}{6}\ (\lambda)^2(\epsilon N - 10) < w|uve_j > z + \lambda\gamma[< v|z >< e_j|u > \ - \ < z|u >< e_j|v >]w \\
&\qquad + \lambda\gamma < z|e_j > [\varepsilon < v|w > u \ - < u|w > v] + \frac{1}{2}\ \gamma[\varepsilon < v|z > ue_j w \\
&\qquad - \varepsilon < z|u > ve_j w - < v|e_j > uzw + \ < e_j|u > vzw] \quad .
\end{aligned}
\tag{2.20}
$$

We first calculate

$$
e_j e_k z = ze_j e_k + 2\lambda < e_k|z > e_j - \lambda < e_j|e_k > z - \lambda < z|e_j > e_k
$$

to find

$$
\begin{aligned}
w(uve^k)(e_j e_k z) &= -\varepsilon(uve^k)w(e_j e_k z) \\
&= -\varepsilon(uve^k)w(ze_j e_k) - 2\lambda(uvz)we_j \\
&\quad + \lambda\epsilon(uve_j)wz + \varepsilon\lambda < z|e_j > (uve^k)we_k \\
&= -\varepsilon\{-\varepsilon(uve^k)(ze_j e_k)w + 2\lambda < w|ze_j e_k > uve^k \\
&\quad - \lambda < uve^k|w > ze_j e_k - \lambda < ze_j e_k|uve^k > w\} \\
&\quad - 2\lambda(uvz)we_j + \lambda\epsilon(uve_j)wz \\
&\quad + \varepsilon\lambda < z|e_j > \{\frac{1}{3}\ \lambda(\epsilon N - 7)uvw + \gamma[< v|w > u - \ < w|u > v]\} \quad .
\end{aligned}
$$

For the first term $(uve^k)(ze_j e_k)w$, we use the Proposition 1 with the replacement $x \to u \to e_j \to e_k$ and $y \to v \to w$ in Eq. (2.11). Also, we note for example

$$
< w|ze_j e_k > uve^k = -uv(ze_j w)
$$

from Eq. (2.1d). Then, after some calculations, we obtain Eq. (2.20).

Step 2

We have the validity of the following relation:

$$
\begin{aligned}
&(xye^j)(uve^k)(e_je_kz) \\
&\quad = \frac{1}{36}\,\varepsilon(\lambda)^2\{(\epsilon N)^2 + 16\epsilon N - 224\}\{xy(uvz) + uv(xyz)\} \\
&\qquad + \frac{1}{18}\,(\lambda)^3(\epsilon N - 10)(\epsilon N - 16) < v|xyu > z \\
&\qquad + \frac{1}{2}\,\gamma\{-uz(xyv) + \epsilon vz(xyu)\} \\
&\qquad - \frac{1}{12}\,\gamma\lambda(\epsilon N - 16)\{\epsilon < y|u > xvz - \varepsilon < u|x > yvz \\
&\qquad\quad - < y|v > xuz + < v|x > yuz\} \\
&\qquad + \frac{1}{6}\,\epsilon\gamma\lambda(\epsilon N - 1)\{< v|z > xyu - < z|u > xyv\} \\
&\qquad + 2\epsilon\gamma\lambda\{< y|z > uvx - < z|x > uvy\} \\
&\qquad + \frac{1}{6}\,\epsilon\gamma\lambda(\epsilon N - 4)\{< y|uvz > x - \epsilon < x|uvz > y\} \\
&\qquad + \gamma\lambda\{< z|xyu > v - < z|xyv > u\} \\
&\qquad + \frac{1}{6}\,\gamma(\lambda)^2(\epsilon N - 10)\{< y|u >< v|x > - < u|x >< v|y >\}z \\
&\qquad + \frac{1}{2}\,\epsilon(\gamma)^2\{[< v|z >< y|u > - < z|u >< y|v >]x \\
&\qquad\quad - [< v|z >< u|x > - < z|u >< v|x >]y\} \quad .
\end{aligned}
\tag{2.21}
$$

To prove it, we first set $w = xye^j$ in Eq. (2.20), and we calculate for example

$$
\begin{aligned}
(uvz)e_j(xye^j) = (xye^j)(uvz)e_j &= x \cdot y \cdot (uvz) + \frac{1}{3}\,\lambda(\epsilon N - 7)xy(uvz) \\
&= \frac{1}{3}\,\lambda(\epsilon N - 7)xy(uvz) + \gamma\{< y|uvz > x - < uvz|x > y\}
\end{aligned}
$$

from Eq. (2.1g) with the replacement $z \to uvz$. Similarly, we evaluate

$$
\begin{aligned}
[(xye^j)uv]e_j z &= -\varepsilon(e_j uv)(xye^j)z = (xye^j)(e_j uv)z \\
&= (xye^j)(uve_j)z - 2\lambda\varepsilon < e_j|v > (xye^j)uz \\
&\quad + \lambda\varepsilon < u|e_j > (xye^j)vz + \lambda\epsilon < v|u > (xye^j)e_j z \\
&= \frac{1}{6}\ \varepsilon\lambda(\epsilon N - 16)(xyu)vz - \frac{1}{6}\ \lambda(\epsilon N - 16)(xyv)uz \\
&\quad + \frac{1}{2}\ \gamma\epsilon\{< y|u > xvz - \ < u|x > yvz\} \\
&\quad - \frac{1}{2}\ \gamma\{< y|v > xuz - \ < v|x > yuz\} \\
&\quad - 2\lambda(xyv)uz + \lambda\varepsilon(xyu)vz \\
&\quad - \lambda < v|u > \{\frac{1}{3}\ \lambda(\epsilon N - 16)xyz + \gamma(< y|z > x - \ < z|x > y)\}
\end{aligned}
$$

from Eqs. (1.23), (2.2), and (2.11).

Inserting these results, we find Eq. (2.21) after some computations, when we utilize also Eq. (1.18).

Step 3

In view of Eqs. (1.13b) and (1.21), we note the validity of

$$
(xye^j)(uve^k)(e_j e_k z) = (uve^j)(xye^k)(e_j e_k z) \quad , \tag{2.22}
$$

which states that it is invariant under $x \leftrightarrow u$ and $y \leftrightarrow v$. The consistency of Eq. (2.21) with Eq. (2.22) can be readily shown to require the validity of

$$
\begin{aligned}
&\gamma\varepsilon\{uv(xyz) - xy(uvz)\} \\
&\quad = \frac{1}{6}\ \gamma\lambda(\epsilon N - 16)\{< z|v > xyu - < u|z > xyv - \ < z|y > uvx + < x|z > uvy \\
&\qquad - < z|uvy > x + \varepsilon < z|uvx > y + < z|xyv > u - \varepsilon < z|xyu > v \\
&\qquad - \varepsilon < y|u > xvz + \epsilon < u|x > yvz + < y|v > xuz - < v|x > yuz\} \\
&\qquad + \frac{1}{2}\ \epsilon(\gamma)^2\{[< v|z >< y|u > - < z|u >< y|v >]x \\
&\qquad - [< v|z >< u|x > - < z|u >< v|x >]y - [< y|z >< v|x > \\
&\qquad - < z|x >< v|y >]u + [< y|z >< x|u > - < z|x >< y|u >]v\} \quad .
\end{aligned} \tag{2.23}
$$

We note especially that all γ-independent terms have disappeared in Eq. (2.23). Therefore, if $\gamma \neq 0$, we must have then

$$
\begin{aligned}
&uv(xyz) - xy(uvz) \\
&\quad = \frac{1}{6}\,\epsilon\lambda(\epsilon N - 16)\{< z|v > xyu - < u|z > xyv - \ < z|y > uvx + < x|z > uvy \\
&\qquad - < z|uvy > x + \varepsilon < z|uvx > y + < z|xyv > u - \varepsilon < z|xyu > v \\
&\qquad - \varepsilon < y|u > xvz + \varepsilon < u|x > yvz + < y|v > xuz - < v|x > yuz\} \qquad (2.24) \\
&\qquad + \frac{1}{2}\,\gamma\{[< v|z >< y|u > - < z|u >< y|v >]x - [< v|z >< u|x > \\
&\qquad - < z|u >< v|x >]y - [< y|z >< v|x > - < z|x >< v|y >]u \\
&\qquad + [< y|z >< x|u > - < z|x >< y|u >]v\} \quad .
\end{aligned}
$$

Setting $y = e_j$ and $z = e^j$ in Eq. (2.24), and summing over j, it gives

$$(\epsilon N - 4)\{\lambda(\epsilon N - 10)uvx - 3\gamma[< x|u > v \ - < v|x > u]\} = 0 \quad . \tag{2.25}$$

Therefore, we must have either $\epsilon N - 4 = 0$ or

$$\lambda(\epsilon N - 10)uvx - 3\gamma[< x|u > v \ - < v|x > u] = 0 \quad . \tag{2.26}$$

Letting $v = e_j$ and $x = e^j$, Eq. (2.26) further leads to

$$(\epsilon N - 1)\{\lambda^2(\epsilon N - 10) + 3\gamma\}u = 0 \quad .$$

However, since $\epsilon N - 1 = 0$ is trivial, this requires

$$\gamma = -\frac{1}{3}\,\lambda^2(\epsilon N - 10) \quad . \tag{2.27}$$

Moreover, if $\gamma \neq 0$, then $\lambda(\epsilon N - 10) \neq 0$ so that Eq. (2.26) together with Eq. (2.27) gives

$$uvx + \lambda[< x|u > v \ - < v|x > u] = 0$$

or equivalently $[u, v, x] = 0$. Conversely, if we have $[x, y, z] = 0$ identically, it is easy to find that $x \cdot y \cdot z$ satisfies Eq. (2.19) with the value of γ being given precisely by Eq. (2.27). Also, we can verify that the case $\epsilon N = 4$ of the quaternion triple system which is not however trivial gives $\lambda = 0$ and $\gamma = -2\alpha$ in the notation of I. This completes the proof of the Proposition 3. ■

3. Solution of the Y–B equation

Here, we will solve the Y–B equation (1.8) in a form given by Eq. (1.25) for either OTS ($\epsilon = +1$) or STS ($\epsilon = -1$) under the additional condition

$$x \cdot y \cdot z = 0 \quad . \tag{3.1}$$

For simplicity, we write

$$P = P(\theta) \quad , \quad P' = P(\theta') \quad , \quad P'' = P(\theta'') \tag{3.2}$$

and similarly for $Q(\theta)$, $R(\theta)$, and $S(\theta)$. Inserting Eq. (1.25) into Eq. (1.8), each side of Eq. (1.8) is exapnded as a sum of 64 terms. The most complicated term is the first one in the expansion of form

$$PP'P''\{(e_j zu)(xye^j)v - (e_j xv)(zye^j)u\}$$

which can be, however, simplified by Eq. (2.11) of the Proposition 2 together with Eqs. (1.13b) and (1.13c). Utilizing Eq. (1.18) and results of the Lemma 1, we find the following expression after somewhat long calculations:

$$
\begin{aligned}
0 &= [v,[u,e_j,z]_{\theta'},[e^j,x,y]_\theta]_{\theta''} - (u \leftrightarrow v, x \leftrightarrow z, \theta \leftrightarrow \theta'') \\
&= K_1\{(yxu)zv - (yzv)xu\} + K_2 < u|v > [x,y,z] + K_3 < x|z > [y,u,v] \\
&\quad + K_4 < u|x > [y,z,v] - \hat{K}_4 < v|z > [y,x,u] \\
&\quad + K_5 < y|z > [x,u,v] - \hat{K}_5 < y|x > [z,v,u] \\
&\quad + K_6 < u|y > [x,z,v] - \hat{K}_6 < v|y > [z,x,u] \\
&\quad + K_7 < z|u > [x,y,v] - \hat{K}_7 < x|v > [z,y,u] \\
&\quad + K_8 < u|[z,y,v] > x - \hat{K}_8 < v|[x,y,u] > z \\
&\quad + K_9 < v|[x,y,z] > u - \hat{K}_9 < u|[x,y,z] > v \\
&\quad + K_{10} < v|y >< u|z > x - \hat{K}_{10} < u|y >< v|x > z \\
&\quad + K_{11} < y|z >< u|v > x - \hat{K}_{11} < y|x >< v|u > z \\
&\quad + K_{12} < y|z >< v|x > u - \hat{K}_{12} < y|x >< u|z > v \\
&\quad + K_{13} < x|z >< y|v > u - \hat{K}_{13} < x|z >< u|y > v
\end{aligned}
\tag{3.3}
$$

where $K_\mu (\mu = 1, 2, \ldots, 13)$ are cubic polynomials of $P,\ Q,\ R,$ and S to be specified below, and $\hat{K}_\mu$ is the same function as K_μ except for the interchange of $\theta \leftrightarrow \theta''$. Note that only $K_1,\ K_2,\ K_3,$ and K_{12} are self-conjugate, i.e. $\hat{K}_\mu = K_\mu$ for $\mu = 1, 2, 3,$ and 12. The explicit expressions for K_μ's are given by

$$
\begin{aligned}
K_1 &= -\frac{1}{6}\ \lambda(\epsilon N - 4)P''P'P + P''R'P - P''P'R - R''P'P \quad , \\
K_2 &= -\frac{1}{3}\ \lambda^2(\epsilon N - 10)P''P'P - \lambda P''P'R - \lambda R''P'P - 2\lambda P''Q'P + P''Q'R + R''Q'P \quad , \\
K_3 &= -\varepsilon K_2 \quad , \\
K_4 &= 0 \quad , \\
K_5 &= -\frac{1}{6}\ \lambda^2(\epsilon N - 10)P''P'P - \frac{1}{3}\ \lambda(\epsilon N - 7)Q''P'P - \lambda P''Q'P - \lambda R''P'P \\
&\quad - Q''P'R + Q''P'S - Q''Q'P + Q''R'P - P''Q'S \quad , \\
K_6 &= -2\lambda P''P'R + R''P'S - P''R'S \quad ,
\end{aligned}
$$

$$
\begin{aligned}
K_7 &= \frac{1}{3}\ \lambda^2(\epsilon N-16)P''P'P+\frac{1}{3}\ \lambda(\epsilon N-16)P''S'P+2\lambda P''P'R \\
&+2\lambda R''P'P+S''P'S+P''S'R+R''S'P-P''S'S-S''S'P \quad , \\
K_8 &= \varepsilon K_6 \quad , \qquad\qquad (3.4) \\
K_9 &= -\hat{K}_5 \quad , \\
K_{10} &= \frac{1}{3}\ \lambda^3(\epsilon N-16)P''P'P+\frac{1}{3}\ \lambda^2(\epsilon N-16)P''S'P+\lambda(P''S'+S''P')R \\
&-\lambda(S''R'-R''S'+S''S')P+\lambda(S''P'-P''S' \\
&+R''P'-P''R')S+R''S'S+S''S'R-S''R'S \\
K_{11} &= -\frac{1}{6}\ \lambda^3(\epsilon N-10)P''P'P+\frac{1}{3}\ \lambda^2(\epsilon N-7)Q''P'P-\lambda^2P''Q'P \\
&+\lambda(Q''Q'+R''Q'-Q''R')P+\lambda(Q''P'-P''Q')S \\
&+R''Q'S-Q''R'S-Q''Q'R \quad , \\
K_{12} &= \frac{1}{6}\ \lambda^3(\epsilon N-10)P''P'P+\lambda^2P''Q'P-\lambda(\epsilon N-1)Q''P'Q-\lambda(S''P'+R''P')Q \\
&+\lambda(P''Q'-Q''P')S+\lambda S''Q'P-\lambda Q''P'R-(\epsilon N)Q''S'Q \\
&-(Q''Q'+Q''R'+R''S'+S''S')Q-(Q''S'-S''Q')S-Q''S'R \quad , \\
K_{13} &= \hat{K}_{11} \quad .
\end{aligned}
$$

We note that if we had had used

$$x\cdot y\cdot z=\gamma\{<y|z>x-<z|x>y\}$$

instead of Eq. (3.1), then K_4, for example, would become

$$K_4=-\frac{1}{2}\ \epsilon\gamma P''P'P$$

instead of zero as in Eq. (3.4). Simlarly, we must add extra term

$$\frac{1}{2}\ \gamma P''P'P$$

to K_6, but not to K_8, so that the relation $K_8 = \varepsilon K_6$ will not hold any longer. For the octonionic triple product, we can further reduce the first term proportional to K_1 by using the identity given in I. Then, we can verify after some calculations that Eqs. (3.3) and (3.4) of the present paper reproduce the corresponding equations in I for $\varepsilon N = 8$, $\lambda = -3\beta$, with $\alpha = \beta^2$.

At any rate, the Yang–Baxter equation will be automatically satisfied, if we have eight equations

$$K_1 = K_2 = K_5 = K_6 = K_7 = K_{10} = K_{11} = K_{12} = 0 \quad . \tag{3.5}$$

First, consider $K_1 = 0$ which can be rewritten as

$$R'/P' - R/P - R''/P'' = \frac{1}{6}\ \lambda(\epsilon N - 4) \quad .$$

However, since $\theta' = \theta + \theta''$, this equation requires the validity of

$$R(\theta)/P(\theta) = a + b\theta \tag{3.6}$$

where we have set

$$a = -\frac{1}{6}\ \lambda(\epsilon N - 4) \tag{3.7}$$

and b is an arbitrary constant. Then, $K_2 = 0$ which can be rewritten as

$$-\frac{1}{3}\ \lambda^2(\epsilon N - 10) - \lambda\left(\frac{R}{P} + \frac{R''}{P''}\right) + \frac{Q'}{P'}\ \left(\frac{R''}{P''} + \frac{R}{P} - 2\lambda\right) = 0$$

can be solved to yield

$$Q(\theta)/P(\theta) = \lambda - \frac{2a\lambda}{2(a-\lambda) + b\theta} \quad . \tag{3.8}$$

Similarly, the condition $K_6 = 0$ which is equivalent to

$$-2\lambda\ \frac{R}{P} + \left(\frac{R''}{P''} - \frac{R'}{P'}\right)\ \frac{S}{P} = 0$$

leads to

$$S(\theta)/P(\theta) = -2\lambda - \frac{2\lambda a}{b\theta} \quad . \tag{3.9}$$

These determine $R(\theta)$, $Q(\theta)$, and $S(\theta)$ in terms of $P(\theta)$. The rest of relations $K_5 = K_7 = K_{10} = K_{11} = K_{12} = 0$ can be then verified to be automatically satisfied after some computations. Rewriting

$$\begin{aligned} [z,x,y]_\theta =& P(\theta)[x,y,z] + A(\theta) < x|y > z \\ & + B(\theta) < z|x > y + C(\theta) < z|y > x \quad , \end{aligned} \tag{3.10}$$

then we find

$$A(\theta) = \{\lambda - \frac{2a\lambda}{2(a-\lambda)+b\theta}\}\ P(\theta) \quad , \tag{3.11a}$$

$$B(\theta) = \{(a-\lambda)+b\theta\}\ P(\theta) \quad , \tag{3.11b}$$

$$C(\theta) = \{-\lambda - \frac{2a\lambda}{b\theta}\}\ P(\theta) \quad , \tag{3.11c}$$

because of Eqs. (1.28). This reproduces Eq. (1.29). As we will see in the next section, the case of the octonionic triple product corresponding to $\varepsilon N = 8$ with the normalization $\lambda = -3\beta = 3$ as in I satisfies the desired condition $x \cdot y \cdot z = 0$. In that case, $a = -2$ by Eq. (3.7) and hence

$$A(\theta)/P(\theta) = \frac{18-3b\theta}{10-b\theta} \quad , \tag{3.12a}$$

$$B(\theta)/P(\theta) = b\theta - 5 \quad , \tag{3.12b}$$

$$C(\theta)/P(\theta) = \frac{12-3b\theta}{b\theta} \quad , \tag{3.12c}$$

which reproduces the result of I as well as that given by de Vega and Nicolai[13)]. This fact serves as a cross–check of our calculations, since these latter computations are based upon entirely different method.

In our derivation of Eq. (3.11), we have implicitly assumed $P(\theta) \neq 0$. However, if we have $P(\theta) = 0$ identically, the situation becomes simpler, since we need then consider only 3 conditions $K_{10} = K_{11} = K_{12} = 0$. Assuming $C(\theta) \neq 0$, the solution is given now by

$$[z,x,y]_\theta = A(\theta) < x|y > z + B(\theta) < z|x > y + C(\theta) < z|y > x \quad , \tag{3.13a}$$

with

$$\frac{B(\theta)}{C(\theta)} = b\theta \quad , \quad \frac{A(\theta)}{C(\theta)} = -\frac{2b\theta}{2b\theta + (\epsilon N - 2)} \tag{3.13b}$$

for arbitrary constant b. Note that the dimension N is completely arbitrary, and that since $P(\theta) = 0$, we need no longer assume here that V is either OTS or STS. Also, Eq. (3.13b) reproduces the result Eq. (4.12) of I for $\varepsilon N = 4$. In this case, we can also forget about the condition $x \cdot y \cdot z = 0$. We remark that Eq. (3.13b) for $\varepsilon = 1$ reproduces the result of Zamolodchikov's so(N) model[14)], while $\varepsilon = -1$ corresponds to sp(N) symmetry.

We may interpret Eq. (3.13a) as the condition $[x, y, z] = 0$ rather than $P(\theta) = 0$. Then, the condition $x \cdot y \cdot z = 0$ can be achieved only if $\epsilon N = 10$ by the Proposition 3. The solution Eqs. (3.11) for $\varepsilon N = 10$ agree, of course, with Eqs. (3.13), when we suitably renormalize λ and b, for example by $a = -\lambda = 1$.

Also if we have $C(\theta) = 0$ in Eq. (3.13a), then the solution of the Y–B equation is rather trivial with $A(\theta) = 0$ and $B(\theta)$ being arbitrary.

Finally, the case $N = 2$ for STS is special, and we can find a more general trigonometric solution in that case, as we will explain in section 5.

4. Condition $x \cdot y \cdot z = 0$

In this section, we seek OTS or STS satisfying the condition $x \cdot y \cdot z = 0$. For the trivial case of $[x, y, z] = 0$ identically, it is, of course, satisfied only for $\varepsilon N = 10$ (i.e. $\varepsilon = 1$ and $N = 10$) by the Proposition 3. Also for the octonionic triple product corresponding to $\varepsilon = 1$ and $N = 8$, we can verify $x \cdot y \cdot z = 0$ by using the result of I. Similarly, for simple cases of $N = 2$ and $N = 4$ for STS to be given shortly, we can directly show the same by explicit computations. However, the task will become increasingly unmangeable for larger values of N.

We can nevertheless give a simple characterization of OTS or STS satisfying $x \cdot y \cdot z = 0$ as follows. For this end, we utilize the method explained in ref. 15 for STS and also briefly in I for OTS. For many STS and OTS, the underlying vector space V is often a module of a Lie algebra L, which we will assume in this section. Also, unless we state otherwise, we assume L to be simple and V to be an irreducible module of L. Let W_1 and W_2 be two L-modules which need not be, however, irreducible. We denote then by $\mathrm{Hom}(W_1 \to W_2)$ be the vector space of all homomorphism from W_1 to W_2, which are compatible with the action of L.

Next, the tensor product $V \otimes V \otimes V$ can be decomposed into a sum of vector spaces with distinct permutation symmetries and we set as in I

$$V = [1] \quad , \quad (V\otimes V\otimes V)_S = [3] \quad , \quad (V\otimes V\otimes V)_A = [1^3] \quad , \quad (V\otimes V\otimes V)_M = [2,1] \quad (4.1)$$

etc, where the suffices S, A, and M refer to the totally symmetric, antisymmetric and mixed symmetries with respect to the permutation group Z_3, and the symbol $[f_1, f_2, f_3]$ with $f_1 \geq f_2 \geq f_3 \geq 0$ designates the standard Young-tableau notation[16)]. Suppose that we have

$$\text{Dim Hom } ([1^3] \to [1]) = 1 \tag{4.2}$$

or

$$\text{Dim Hom } ([3] \to [1]) = 1 \quad , \tag{4.3}$$

where Dim W is the dimension of vector space W. This implies[15)], then, the existence of unique L–covariant triple product $[x, y, z]$ in V which is totally antisymmetric for the former, or totally symmetric for the latter. Moreover, these products can be shown to satisfy the axioms of OTS or STS, if some additional conditions such as

$$\text{Dim Hom } ([4,1] \to [1]) \leq 2$$

etc. hold valid. However, since these are discussed in detail in ref. 15 and also in I, we will not go into detail.

Returning now to the dotted product $x \cdot y \cdot z$, the validity of Eqs. (2.3a) and (2.3b) implies that $x \cdot y \cdot z$ is contrarily an element of

$$\text{Hom } ([2,1] \to [1]) \quad .$$

Suppose that we have

$$\text{Dim Hom } ([2,1] \to [1]) \leq 1 \tag{4.4}$$

in addition. Then, following the reasoning of ref. 15, $x \cdot y \cdot z$ can be rewritten in the form of Eq. (2.19) for some constant γ, since $< y|z > x - \ < z|x > y$ can be easily seen also to be an element of Hom$([2,1] \to [1])$. In that case, the Proposition 3 guarantees $x \cdot y \cdot z = 0$ identically, provided that the original OTS or STS is not trivial, i.e. $[x, y, z] \neq 0$ with $\varepsilon N \neq 4$. Therefore, we have only to verify the validity of Eq. (4.4). For OTS, both octonionic ($N = 8$) and Malcev ($N = 7$) triple products have been shown in I to satisfy the condition (4.4) so that we have $x \cdot y \cdot z = 0$ identically. As we have already stated, this can be easily verified also by a direct computation for the former case of $N = 8$. Therefore, two cases of $N = 7$ and 8 with $\varepsilon = 1$ furnish solutions of the Y–B equation. Although there may exist other OTS satisfying the condition, we do not know yet. Also the case of $N = 4$ corresponding to the quaternionic triple product does not satisfy $x \cdot y \cdot z = 0$. However, we have already found the solution for this case in I by a different method.

We will devote the rest of this section to the STS case of $\varepsilon = -1$. It is now known that there exists intimate inter–relationship between a simple Lie algebra other than A_1 and a STS. Especially, Asano[17)] shows that we can construct any simple Lie algebra L_0 from some STS, and conversely that a STS can always be constructed from any simple Lie algebras L_0 other than A_1. Therefore, we can construct STS's from any simple Lie algebras, following the method of the ref. 17. However, as we will show shortly, only STS's constructed from $L_0 = A_2,\ G_2,\ F_4,\ E_6,\ E_7,$ and E_8 satisfy the desired condtion $x \cdot y \cdot z = 0$. Let H_0 and $\alpha \ \epsilon\ \Delta$ be respectively the Cartan sub-algebra in Chevalley basis and non-zero root of a complex simple Lie algebra L_0, where Δ is the root-system with respect to some lexicographical ordering[18)]. Let ρ be the highest root normalized to

$$(\rho, \rho) = 2 \quad . \tag{4.5}$$

Setting

$$h = [E_\rho, E_{-\rho}] \quad , \tag{4.6}$$

we can decompose L_0 into a direct sum

$$L_0 = V_2 \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus V_{-2} \tag{4.7}$$

which satisfies moreover

$$[h, x_n] = n x_n \quad , \quad \text{if} \quad x_n \,\epsilon\, V_n \quad , \tag{4.8a}$$

and

$$[V_n, V_m] \subset V_{n+m} \tag{4.8b}$$

for $n, m = 0,\ \pm 1,$ and $\pm\, 2$. More explicitly,

$$\begin{aligned} V_2 &= \{x | x = cE_\rho\} \quad , \quad V_{-2} = \{x | x = cE_{-\rho}\} \quad , \\ V_1 &= \{x | x = \sum_\alpha c_\alpha E_\alpha, (\rho, \alpha) = 1\} \quad , \\ V_{-1} &= \{x | x = \sum_\alpha c_\alpha E_\alpha, (\rho, \alpha) = -1\} \quad , \\ V_0 &= \{x | x = x_0 + \overline{x} \quad , \quad x_0 \epsilon H_0 \quad , \quad \overline{x} = \sum_\alpha c_\alpha E_\alpha \text{ with } (\rho, \alpha) = 0\} \end{aligned} \tag{4.9}$$

for constants c, and c_α's.

We identify our module V to be

$$V = V_1 \tag{4.10}$$

by a reason to be given shortly. First for any $x,\ y \,\epsilon\, V_1$, there exists an inner product $< x | y >$ defined by

$$[x, y] = 2 < x | y > E_\rho \tag{4.11}$$

since the left side must belong to the space V_2 by Eq. (4.8b). Clearly, $< x | y >$ is non-degenerate in V_1 and satisfies the symplectic condition

$$< x | y > = - < y | x > \quad . \tag{4.12}$$

Following Asano, we then introduce a triple product xyz in $V = V_1$ by

$$xyz = \frac{1}{2} \ \{[z,[y,[x,E_{-\rho}]]] + [z,[x,[y,E_{-\rho}]]]\} \quad . \tag{4.13}$$

The fact that $xyz \ \epsilon \ V_1$ follows again from Eq. (4.8b). It is easy to show the validity of

$$xyz = yxz \tag{4.14a}$$

$$xyz - xzy = 2 < y|z > x \ - < x|y > z - \ < z|x > y \quad . \tag{4.14b}$$

Finally, Asano[17)] also proves the derivation relation

$$uv(xyz) = (uvx)yz + x(uvy)z + xy(uvz) \tag{4.15}$$

for the product. Therefore, the product xyz defines a symplectic triple system with λ normalized to be 1.

Actually, $V = V_1$ can be regarded as a module of a sub-Lie algebra L of L_0. In general, L is a reductive Lie algebra, i.e., it is a direct sum of semi-simple and Abelian algebras. However, for our purpose, we regard it as a simple Lie algebra by choosing only its maximal simple algebra and deleting all others contained therein. The Dynkin diagram of L is precisely the one obtained from the general Dynkin diagram of L_0, by deleting simple roots connected with the lowest root $-\rho$. We are now in a position to characterize L, V, and $N =$ Dim V for any L_0. First for any classical Lie algebras $L_0 = A_n (n \geq 3)$, $B_n (n \geq 2)$, $C_n (n \geq 2)$ and $D_n (n \geq 3)$, we find respectively

$$L = A_{n-2}, B_{n-1}, C_{n-1}, \text{ and } D_{n-1} \quad .$$

However the condition $x \cdot y \cdot z = 0$ is not satisfied by any of these by the following reason. For $L = C_{n-1}$, V is a $N = 2(n-1)$ dimensional irreducible module of L. However, the resulting STS turns out to be trivial. On the other sides, we find that V's for $L = A_{n-2}$, B_{n-1}, and D_{n-1} are reducible and that the condition Eq. (4.4) is not satisfied. The exception is for $L_0 = A_2$ where L is null with $N = 2$. In this case, we can verify $x \cdot y \cdot z = 0$ by a direct computation. Next, consider 5 exceptional Lie algebras $L_0 = G_2,\ F_4,\ E_6,\ E_7,$ and E_8 where we find

$$\begin{aligned}
&\text{(i)}\ \ L_0 = G_2 \quad , \quad L = A_1 \quad , \quad N = 4 \quad , \\
&\text{(ii)}\ \ L_0 = F_4 \quad , \quad L = C_3 \quad , \quad N = 14 \quad , \\
&\text{(iii)}\ \ L_0 = E_6 \quad , \quad L = A_5 \quad , \quad N = 20 \quad , \\
&\text{(iv)}\ \ L_0 = E_7 \quad , \quad L = D_6 \quad , \quad N = 32 \quad , \\
&\text{(v)}\ \ L_0 = E_8 \quad , \quad L = E_7 \quad , \quad N = 56 \quad .
\end{aligned} \tag{4.16}$$

Moreover, V for all these cases are found to be irreducible L-modules, satisfying the desired condition

$$\text{Dim Hom } ([2,1] \to [1]) = 1$$

so that we have $x \cdot y \cdot z = 0$ automatically for all these cases. We have also explicitly verified it for the simplest case of $N = 4$ for $L = A_1$. The fact that 5 cases listed in Eq. (4.16) defines indeed STS's can be also directly shown by the method given in ref. 15.

In conclusion, 6 cases of $N = 2,\ 4,\ 14,\ 20,\ 32,$ and 56 with $\varepsilon = -1$ furnish solutions of the Y–B equation. In this connection, we note that both irreducible module V for $L = A_5$ and C_3 in Eq. (4.16) correspond to totally antisymmetric Young tableau $[1^3]$ of su(6) and sp(6) with respect to their 6–dimensional fundametal representations, while the 32–dimensional module for $L = D_6$ is its basic spinor representation. The case of $L = E_7$ with $N = 56$ refers, of course, to the Freudenthal's triple system.

Finally, $N = 4$ for $L = A_1$ is realizable as the 4–dimensional spin 3/2 representation of the so(3) $\simeq$ su(2) algebra. Let x_M with $M = 2m = 3,\ 1,\ -1,\ -3$ be its basis, corresponding to the magnetic quantum number $m = 3/2,\ 1/2,\ -1/2,$ and $-3/2$, respectively. We may normalize them according to

$$< x_M | x_{M'} > = -\frac{M}{2}\ \delta_{M+M',0} \quad . \tag{4.17}$$

Then, the totally symmetric triple product $[x, y, z]$ satisfies

$$[x_{M_1}, x_{M_2}, x_{M_3}] = C(M_1, M_2, M_3) x_{M_1+M_2+M_3} \tag{4.18}$$

for a constant $C(M_1, M_2, M_3)$ where we set $x_M = 0$ identically, unless we have $M \neq 3,\ 1,\ -1,$ or -3. The physical meaning of C_{M_1,M_2,M_3} is that it is precisely the Clebsch–Gordan coefficient of totally symmetric tensor product $(V \otimes V \otimes V)_S$ into the unique spin 3/2 representation $V = \{3/2\}$. Indeed, we calculate

$$(V \otimes V \otimes V)_S = \{9/2\} \oplus \{5/2\} \oplus \{3/2\}$$

where $\{j\}$ designates the irrreducible module of su(2) with spin $j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \ldots$. Non-zero Clebsch–Gordan coefficients $C(M_1, M_2, M_3)$ are tabulated below to be

$$\begin{gathered}
C(1,1,1) = C(-1,-1,-1) = \frac{2}{3} \quad , \\
C(-1,-1,1) = -C(-1,1,1) = \frac{1}{3} \quad , \\
C(-1,-1,3) = C(1,1,-3) = -2 \quad , \\
C(-1,1,-3) = C(-1,-3,3) = -\frac{1}{2} \quad , \\
C(-1,1,3) = C(1,-3,3) = \frac{1}{2} \quad , \\
C(-3,3,3) = -C(-3,-3,3) = 3 \quad .
\end{gathered} \tag{4.19}$$

All other cases except for permutations of $M_1,\ M_2,$ and M_3 in the above list give zero value for $C(M_1, M_2, M_3)$. We note that $C(M_1, M_2, M_3)$ is totally symmetric in $M_1,\ M_2,$ and M_3. Moreover it admits an automorphism $\sigma\ :\ V \to V$

$$\sigma(x_M) = \varepsilon(M) x_{-M} \quad , \tag{4.20a}$$

$$\varepsilon(M) = \begin{cases} 1 \quad , & \text{for} \quad M > 0 \\ -1 \quad , & \text{for} \quad M < 0 \end{cases} \tag{4.20b}$$

which satisfies

$$\sigma([x,y,z]) = [\sigma(x), \sigma(y), \sigma(z)] \tag{4.21a}$$

$$< \sigma(x) | \sigma(y) > = < x | y > \tag{4.21b}$$

$$\sigma^2 = -I \quad . \tag{4.21c}$$

The $N = 4$ triple product possesses $N = 2$ sub-algebra consisting of two element x_3 and x_{-3}. One interesting aspect of the STS for $N = 2$ is that the following special identity holds valid for the normalization condition $\lambda = 1$ with $< x_{-3}|x_3 > = 3/2$:

$$
\begin{aligned}
&[x, y,[u, v, z]] - [u, v, [x, y, z]] \\
&\quad = < x|u > [v, y, z] + < y|v > [u, z, x] + < x|v > [u, y, z] \\
&\quad + < y|u > [v, z, x] + [< y|u >< z|v > + < y|v >< z|u >]x \\
&\quad + [< z|u >< x|v > + < z|v >< x|u >]y + [< x|v >< z|y > \\
&\quad + < y|v >< z|x >]u + [< z|x >< y|u > + < z|y >< x|u >]v
\end{aligned} \tag{4.22}
$$

as we may verify easily. Because of Eq. (4.22), we can reduce $(yxu)zv - (yzv)xu$ in terms of simpler expressions. Together with other identities for $\varepsilon N = -2$, this enables for us to considerably simplify Eq. (3.3) further, and we can find a more general solution for the Y–B equation. This will be explained in detail in the next section.

5. Solution of the STS for N = 2

For the special case of $N = 2$ for STS, we can find a more general solution than the one given by Eqs. (1.29). The reason is first because we have the special relation Eq. (4.22) for that case. Second, we note the validity of the identity

$$
< y|z > x + < z|x > y + < x|y > z = 0 \tag{5.1}
$$

for $N = 2$ with $\varepsilon = -1$ since the left side of Eq. (5.1) is totally antisymmetric in $x,\ y,$ and z. Especially, $[z, x, y]_\theta$ given by Eq. (1.25) is invariant under the transformation

$$
Q(\theta) \to Q(\theta) + F(\theta) \quad , \quad R(\theta) \to R(\theta) + F(\theta) \quad , \quad Q(\theta) \to Q(\theta) + F(\theta) \quad , \tag{5.2}
$$

for an arbitrary function $F = F(\theta)$. However the expression Eq. (3.3) is not manifestly invariant under it, implying that we may reduce the expression further into a simpler form.

Utilizing Eq. (5.1), we can moreover note

$$
< y|z >< u|v > = - < y|u >< v|z > - < y|v >< z|u > \quad , \tag{5.3a}
$$

$$< u|[x,y,z] > v \ - < v|[x,y,z] > u = < u|v > [x,y,z] \quad , \tag{5.3b}$$

$$< u|v > [x,y,z] = - < v|y > [x,u,z] - \ < y|u > [x,v,z] \tag{5.3c}$$

and so on. Using these relations as well as Eq. (4.22), we can, now, simplify the right side of Eq. (3.3) as

$$\begin{aligned}
0 &= [v,[u,e_j,z]_{\theta'},[e^j,x,y]_\theta]_{\theta''} - (u \leftrightarrow v, x \leftrightarrow z, \theta \leftrightarrow \theta'') \\
&= W_1 < v|y > [u,x,z] - \hat{W}_1 < u|y > [v,z,x] \\
&\quad + W_2 < x|y > [u,z,v] - \hat{W}_2 < z|y > [v,x,u] \\
&\quad + W_3\{< u|[z,y,v] > x + < v|[x,y,u] > z \\
&\quad + \ < v|[x,y,z] > u + \ < u|[x,y,z] > v\} \\
&\quad + W_4 < v|y >< u|z > x - \hat{W}_4 < u|y >< v|x > z \\
&\quad + W_5 < y|u >< v|z > x - \hat{W}_5 < y|v >< u|x > z \\
&\quad + W_6 < v|x >< z|u > y
\end{aligned} \tag{5.4}$$

where we find

$$\begin{aligned}
&W_1 = 3K_1 - K_2 - \hat{K}_4 - \hat{K}_6 + \hat{K}_7 + \frac{1}{4}\ (K_8 - \hat{K}_8) + \frac{1}{4}\ (K_9 + 3\hat{K}_9) \quad , \\
&W_2 = -3K_1 + K_3 - K_4 + \hat{K}_5 - \hat{K}_7 - \frac{1}{4}\ (3K_8 + \hat{K}_8) + \frac{1}{4}\ (K_9 - \hat{K}_9) \quad , \\
&W_3 = \frac{1}{4}\ (K_8 - \hat{K}_8 + K_9 - \hat{K}_9) \quad , \\
&W_4 = 3K_1 + K_{10} - K_{11} + \hat{K}_{12} - K_{13} \quad , \\
&W_5 = -K_{11} + \hat{K}_{13} \quad , \\
&W_6 = -K_{12} + \hat{K}_{12} \quad .
\end{aligned} \tag{5.5}$$

However, in view of relations such as $K_{11} = \hat{K}_{13},\ K_4 = 0,\ K_3 = K_2,\ K_8 = -K_6$, and $K_9 = -\hat{K}_5$ as well as $\hat{K}_{12} = K_{12}$ by Eq. (3.4), we have identities

$$W_5 = W_6 = 0 \quad , \quad W_1 + W_2 + 2W_3 = 0 \quad , \tag{5.6}$$

so that the Y–B equation is satisfied, provided that we have 3 equations;

$$W_1 = W_3 = W_4 = 0 \quad . \tag{5.7}$$

Moreover, we can verify the fact that W_1, W_3, and W_4 are invariant under the transformation Eq. (5.2). Therefore, choosing $F = -Q$, we can effectively set $Q = 0$ so that

$$[z,x,y]_\theta = P(\theta)xyz + R(\theta) < z|x > y + S(\theta) < y|z > x \quad . \tag{5.8}$$

First consider the relation $W_3 = 0$ which can be rewritten as

$$(3 + S/P)(R'/P' - R''/P'') = (3 + S''/P'')(R'/P' - R/P) \quad , \tag{5.9}$$

assuming $P(\theta) \neq 0$ with $\lambda = 1$. Then, $W_1 = 0$ together with $W_3 = 0$ leads similarly to the validity of

$$\begin{aligned} &(3 + S/P)(R'/P' - R''/P'') + (3 + S'/P')(R/P + R''/P'') \\ &\quad + (3 + S''/P'')(3 + S/P) - (3 + S'/P')(3 + S/P) - (3 + S'/P')(3 + S''/P'') = 0 \end{aligned} \tag{5.10}$$

while $W_4 = 0$ is rewritten as

$$\begin{aligned} &-9P''P'P - 6P''S'P + 3(P''R'P - P''P'R - R''P'P) \\ &\qquad + P''S'R + S''P'R - S''R'P + R''S'P - S''S'P + S''P'S - P''S'S \\ &\qquad + R''P'S - P''R'S + R''S'S + S''S'R - S''R'S = 0 \quad . \end{aligned} \tag{5.11}$$

There exist two distinct classes of solutions. First, we note that Eqs. (5.9), (5.10), and (5.11) admit a solution where we have

$$S(\theta) = -3P(\theta) \tag{5.12}$$

while $R(\theta)$ remains arbitrary. Then, noting

$$xyz = [x,y,z] + \ < y|z > x - \ < z|x > y \quad ,$$

this gives the first solution:

$$[z,x,y]_\theta = P(\theta)\{[x,y,z] - 2 < y|z > x\} + T(\theta) < z|x > y \quad , \tag{5.13}$$

where both $P(\theta)$ and $T(\theta)$ are arbitrary functions of θ. The old solution Eqs. (1.29) with $\lambda = a = 1$ corresponds to a special choice of

$$T(\theta) = \left(-1 + b\theta + \frac{2}{b\theta} \right) P(\theta) \quad .$$

However, since $T(\theta)$ is now arbitrary, we can set $T(\theta) = 0$ in Eq. (5.13), if we wish.

The second solution is, in constrast, of trigonometric type with

$$\begin{aligned} R(\theta)/P(\theta) &= 3 \quad , \\ S(\theta)/P(\theta) &= -3 + \frac{6}{1 - \exp(k\theta)} \end{aligned} \tag{5.14}$$

where k is an arbitrary constant. Then, the solution is given by

$$\begin{aligned} [z, x, y]_\theta =& P(\theta)\{[x, y, z] + 2 < z|x > y \\ &+ \left(-2 + \frac{6}{1 - \exp(k\theta)} \right) < y|z > x\} \quad . \end{aligned} \tag{5.15}$$

Another constant solution can be obtained from this also by letting $k \to -\infty$ for $\theta > 0$ to give

$$[z, x, y]_\theta = P(\theta)\{[x, y, z] + 2 < z|x > y + 4 < y|z > x\} \quad . \tag{5.16}$$

We can directly verify that this is also a solution. Similarly, for $k \to +\infty$, Eq. (5.15) will give a special case of Eq. (5.13).

Note that we cannot here change arbitrarily the normalizations of triple products and inner product since they must satisfy the condition Eq. (4.22).

6. Concluding Remarks

In this paper as well as in the preceeding one, we have found several solutions of the Yang–Baxter equation in a triple product form under the ansatz Eq. (1.9). First, we note that we can dispense with the symmetry condition now by introducing the θ–dependent triple products $[x, y, z]_\theta$ and its conjugate $[x, y, z]^*_\theta$ by

$$R^{cd}_{ab}(\theta) = < e^c|[e^d, e_a, e_b]_\theta > = < e^d|[e^c, e_b, e_a]^*_\theta > \quad . \tag{6.1}$$

Then, the Yang–Baxter equation (1.6) is completely equivalent to the validity of the triple product equation;

$$
\begin{aligned}
&[v,[u,e_j,z]_{\theta'},[e^j,x,y]_\theta]^*_{\theta''} \\
&\quad=[u,[v,e_j,x]^*_{\theta'},[e^j,z,y]^*_{\theta''}]_\theta
\end{aligned}
\tag{6.2}
$$

without any additional ansatz. Note that Eq. (6.2) is invariant under

$$
u\leftrightarrow v,\quad x\leftrightarrow z,\quad \theta\leftrightarrow\theta'',\quad [x,y,z]\leftrightarrow[x,y,z]^* \quad . \tag{6.3}
$$

Especially, we need not even assume that the inner product $< x|y >$ is either symmetric or antisymmetric as we have done in the present note. If $[x,y,z]^*_\theta=[x,y,z]_\theta$, then Eq. (6.2) reduces to Eq. (1.8). We will make an attempt to solve the general equation (6.2) in the future with possible uses of more general triple systems other than OTS and STS considered in this note.

Finally, it may be worthwhile to briefly sketch a history of uses of triple products in theoretical physics. It appears that Y. Nambu[19)] was the first person to have suggested a possible generalization of Heisenberg equation of motions in the quantum mechanics by introducing some triple products. Also, I. Bars[20)] has attempted to use triple systems to be somehow related to sub–constituent blocks of quarks and leptons. On the other side, Truini and Biedenharn[21)] have utilized the so–called Jordan–pair system (which is somewhat related to our Freudenthal's triple system) in their model of constructing a grand–unified theory. More recently, Günaydin and co–workers[22)] have works utilizing ternary algebras for constructions of superconformal algebras.

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